UG THESIS REPORT

ON

PRIMORDIAL NON GAUSSIANITY AS A WINDOW TO NEW AGE COSMOLOGY

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MAY 2024

Primordial Non Gaussianity as a window to New Age Cosmology

UG THESIS REPORT

Submitted in partial fulfillment of the requirements for the award of the degree

BACHELOR OF TECHNOLOGY

in

MECHANICAL ENGINEERING

Submitted by:

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Ahmedabad University

May 2024

CANDIDATE'S DECLARATION

I hereby declare that the project entitled "**Primordial Non Gaussianity as a window to New Age Cosmology**" submitted in partial fulfillment for the award of the degree of Bachelor of Technology in Mechanical Engineering completed under the supervision of **Professor Pankaj S. Joshi, Director ICSC**, Ahmedabad University is an authentic work.

Further, I declare that I have not submitted this work for the award of any other degree elsewhere. I certify that whenever I have used materials (data, images, theoretical analysis, and text) from other sources, I have given full credit to them in the text of the report and giving their details in the references.

Signature and name of the student with date:

CERTIFICATE by Project Guide

It is certified that the above statement made by the student is correct to the best of my knowledge.

Signature: ______________________

Date: ______________________

Designation: ______________________

ACKNOWLEDGMENTS

I would like to express my earnest gratitude towards my advisor, Dr. Pankaj S. Joshi, for his unwavering guidance and support throughout the thesis. His insights and questions were invaluable in making me question deeper towards the fundamentals and also helped me overcome challenges throughout the process. I am also very grateful for the time he invested in me, which helped me develop my skills as a researcher.

My friends at ICSC have inspired me in many ways along with their constant moral support throughout my thesis. The amount of dedication they have shown towards their field is simply remarkable and I aspire to be like them in the future. I am very fortunate to be a part of their community. I would like to specially thank Oem Trivedi for his immense help during the thesis, which allowed me to clear my fundamentals and gave me a feel for the subject. I would also like to thank Kharanshu Solanki for helping me with the figures of the thesis.

I would like to dedicate this thesis to my parents and grandparents. The thesis would not have been possible without their constant love, support, and blessings. I would also like to thank my family who have provided me with constant support and well wishes. Without their help, I would not have been able to achieve my goals.

I would also like to thank all my friends and colleagues for encouraging me to follow my dreams. Their presence in my life has made the journey towards completing this thesis a very fulfilling journey.

Lastly, I would like to thank god for giving me the privilege to follow my dreams.

ABSTRACT

Our current comprehension of the Universe is shaped by precise measurements of structures within the *Cosmic Microwave Background* (CMB) and the arrangement and forms of galaxies that outline the Universe's *Large-Scale structure* (LSS). A fundamental assumption underlying cosmological observations is that the field responsible for the observed structures is initially Gaussian with extremely high accuracy. However, a slight departure from Gaussian behaviour represents a consistently robust theoretical prediction in models that elucidate the observed Universe; this departure is inherently present even in the most basic scenarios. Moreover, a majority of inflationary models yield significantly elevated levels of non-Gaussian behaviour, and hence, it provides a direct window into the early Universe's dynamics. Its detection would mark a groundbreaking revelation in cosmology, offering insights into physics at near GUT scales. This study aims to understand the key features of Primordial Non-Gaussianity and its relationship with inflation. In particular, it utilizes statistical methods, including Monte Carlo Markov Chains and Chi-Square Analysis, to apply constraints from Primordial Non-Gaussianity to assess the accuracy and validity of various inflation models.

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UNITS AND NOTATION

For most of the thesis, the adopted signature is $(-,+,+,+)$. The thesis also employs the units, where fundamental constants are set to 1:

$$\hbar = c = k_b = 1.$$

In these units, there is only one fundamental dimension, energy, and the dimensions of other quantities are derived accordingly:

$$[\text{Energy}] = [\text{Mass}] = [\text{Temperature}] = [\text{Length}]^{-1} = [\text{Time}]^{-1}.$$

In the table below I have listed some important constants and their conversion factors in different units.

Table 1. Constants and Conversion Factors

Quantity	**Value**	**Equivalent**
1 GeV^{-1}	1.97×10^{-14} cm	6.59×10^{-25} sec
1 Mpc	3.08×10^{24} cm	1.56×10^{38} GeV^{-1}
M_{Pl}	1.22×10^{19} GeV	-
ρ_c	$1.87h^2 \times 10^{-29}$ g cm^{-3}	$1.05h^2 \times 10^4$ eV cm^{-3} $8.1h^2 \times 10^{-47}$ GeV^4
T_0	2.75 K	2.3×10^{-13} GeV
G	6.674×10^{-11} m^3 kg^{-1} s^{-2}	-

INTRODUCTION

Einstein's revolutionary theory of general relativity in 1915 sparked a new era of a paradigm shift in physics where we could model the universe and its evolution in terms of physical laws. With the discovery of other galaxies by Edwin Hubble in 1923 a new field of research opened. In 1922-1924 Alexander Friedman gave the first dynamical metric taking the scale factor and the cosmological constant in consideration [74, 75]. In 1927, Georges Lemaître independently discovered the same metric, which was subsequently translated into English in 1931 [109, 110]. Following this Howard Robertson and Arthur Walker in 1935-1936 rigorously proved that the Friedman Lemaitre Robertson Walker (FLRW) metric is the only metric which satisfies the condition for a homogeneous and isotropic universe [151–153]. In 1946 George Gamov gave the first work which talked about the hot and dense early universe from the theory of nucleosynthesis [76]. Alpher and Herman hypothesized the existence of a relic radiation background of the order of a few Kelvin in 1948 [9, 10]. In 1965, Penzias and Wilson found the first record of the cosmic microwave background(CMB) which gave rise to the hot big bang model which is also referred to as the standard model of modern cosmology [148].

STANDARD MODEL OF BIG BANG COSMOLOGY

The standard big bang cosmology model describes the evolution of the early universe to a great extent and is our best working theory for the evolution of the universe. The constituents of the universe dominate the evolution of the universe depending on the time scale and the associated temperature. The early universe is radiation dominated which shifted to matter dominated after sufficient lowering of the temperature at later times. The model however has some unsolved cosmological problems in this scenario. First we will mathematically model the standard big-bang model and then discuss the associated problem arising after the formulation.

From the standard cosmological principle [115], for a homogeneous and isotropic universe at large scales, the metric takes the form:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1}$$

Here the values of k = 1,0, and -1 represent a closed (spherical), flat, and open (hyberbolic) universe respectively. This is derived by calculating the metric for three different cases and then doing a coordinate transformation to reach the common metric. Now to get the dynamics for the spacetime, we solve the Einstein Field equations for the FLRW metric. The Einstein Field Equations are given as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{2}$$

Here the $R_{\mu\nu}$, R, $T_{\mu\nu}$, and G are the Ricci tensor, Ricci scalar, Energy Momentum tensor, and gravitational constant respectively. Here Λ is the cosmological constant originally introduced by Einstein.

Solving the Einstein Field equations for the FLRW metric yields the Friedmann equations. These equations describe the dynamics of the entire universe in terms of the scale factor $a(t)$ with t defined as the cosmic time. Firstly we define the Christoffel symbols or the connection coefficients which can be intuitively understood as the parallel transport of one basis vector of the space time manifold along other basis vectors.

$$\Gamma^{\mu}_{\ \nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(\frac{\partial g_{\rho\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\rho\lambda}}{\partial x^{\nu}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\rho}}\right) \tag{3}$$

The non vanishing components for the Christoffel symbols for the FLRW metric are given as

$$\begin{array}{lll} \Gamma^{t}{}_{rr} = -a\dot{a}\left(\frac{1}{1-kr^2}\right) & \Gamma^{t}{}_{\theta\theta} = -a\dot{a}\,r^2 & \Gamma^{t}{}_{\phi\phi} = -a\dot{a}\,r^2\sin^2\theta \\ \Gamma^{r}{}_{rr} = -\frac{kr}{1-kr^2} & \Gamma^{r}{}_{tr} = \frac{\dot{a}}{a} & \Gamma^{r}{}_{\theta\theta} = r(1-kr^2) \\ \Gamma^{r}{}_{\phi\phi} = r\sin^2\theta\ (1-kr^2) & \Gamma^{\theta}{}_{\phi\phi} = \sin\theta\cos\theta & \Gamma^{\theta}{}_{t\theta} = \Gamma^{\theta}{}_{\theta t} = \frac{\dot{a}}{a} \\ \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r} & \Gamma^{\phi}{}_{t\phi} = \Gamma^{\phi}{}_{\phi t} = \frac{\dot{a}}{a} & \Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r} \\ \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta & & \end{array} \tag{4}$$

Similarly we can calculate the Riemann Tensor

$$R^{\lambda}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\lambda}_{\sigma\nu} - \partial_\nu \Gamma^{\lambda}_{\sigma\mu} + \Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\mu\sigma} \tag{5}$$

Since the Riemann Curvature doesn't explicitly appear in the Einstein equations, we then calculate the Ricci Tensor by taking the contraction of the first index with the third index of the Riemann Curvature Tensor. The non vanishing components of the Ricci Tensor are given as follows:

$$\begin{aligned} R_{tt} &= -3\ \frac{\ddot{a}}{a} \\ R_{rr} &= \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\ \left(\frac{1}{1-kr^2}\right) \\ R_{\theta\theta} &= \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\ r^2 \\ R_{\phi\phi} &= \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\ r^2\ sin^2\theta \end{aligned} \tag{6}$$

The Ricci Scalar is the trace of the Ricci Tensor and can be calculating by contracting the Ricci Tensor with $g^{\mu\nu}$ which is given as:

$$R = \frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2 + k\right), \tag{7}$$

Finally calculating the Einstein Tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{8}$$

We get the non vanishing solutions

$$\begin{aligned}
G_{tt} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \\
G_{rr} &= -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)\left(\frac{1}{1-kr^2}\right) \\
G_{\theta\theta} &= -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) r^2 \\
G_{\phi\phi} &= -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) r^2 \, sin^2\theta
\end{aligned} \tag{9}$$

Solving the Einstein equations for the FLRW metric, we get the Friedmann equations. The two Friedmann Equations for the FLRW metric are given as

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G\rho + \Lambda}{3} \tag{10}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{11}$$

We can infer from the symmetries present in the metric that the total stress-energy tensor should be diagonal. An additional effect of isotropy leads the spatial components to be equal. Therefore we find a stress energy of this form:

$$T^{\mu}_{\nu} = (\rho + P)U^{\mu}U_{\nu} + \, Pg^{\mu}_{\ \nu} = \mathrm{diag}(-\rho, P, P, P) \tag{12}$$

This is also exactly the stress-energy tensor for a perfect fluid. Therefore we can take the universal fluid under the barytropic assumption, i.e., a fluid whose pressure depends only on its density, $P \equiv P(\rho)$. Subsequently the equation of state is formulated as:

$$P = w\rho \tag{13}$$

For non relativistic particles, we have no pressure and therefore $w = 0$. This type of matter is called as dust. Now taking the trace of the momentum tensor

$$T^{\mu}_{\mu} = -\rho + 3P \tag{14}$$

For relativistic particles, the stress energy tensor is traceless and therefore $w = 1/3$. Now we will look at the conservation laws in connection to the energy-momentum tensor. We know that for Minkowski space, momentum and energy are conserved. The law of energy conservation gives us the continuity equation which shows that the rate of change of the density equals the rate of divergence of the energy flux.

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho u) = 0 \tag{15}$$

where we have taken u as the velocity of the fluid. Similarly from the law of conservation of momentum gives us the Euler equation

$$\rho \frac{du}{dt} = \rho \left(\frac{\partial}{\partial t} + u.\nabla \right) u = -\nabla P \tag{16}$$

These conservation laws are then combined to yield a four component conservation equation for the energy-momentum tensor

$$\nabla_{\mu} T^{\mu\nu} = 0. \tag{17}$$

From the conservation law, the spatial components give

$$\nabla_{\mu} T^{\mu i} = \nabla_0 T^{0i} + \nabla_j T^{ji} = 0 + \nabla_j T^{ji} = P \nabla_j g^{ij} = 0, \tag{18}$$

here the last step equals to zero due to the property of the metric being covariantly conserved. Therefore only the conservation law for the zeroth component is of interest

$$\nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\nu} T^{\nu 0} + \Gamma^0_{\mu\nu} T^{\mu\nu} = 0 \tag{19}$$

which for a perfect fluid becomes

$$\dot{\rho} + \Gamma^\mu_{\mu 0}\rho + \Gamma^0_{00}\rho + \Gamma^0_{ij} T^{ij} = 0. \tag{20}$$

From the the Christoffel symbols which we derived earlier, from Eq. (4), we get

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{21}$$

which is the continuity equation, which can be also written in the following form,

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{22}$$

Integrating the above equation, we get

$$\rho \propto a^{-3(1+w)} \tag{23}$$

We can also write Eq. (10) in terms of the Hubble Parameter,

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \tag{24}$$

Now we can write this in terms of the critical density where the subscript '0' denotes the quantities which are evaluated in the present time at $t = t_0$. Assuming a flat universe ($k = 0$) we have the following critical density.

$$
\begin{aligned}
\rho_{c,0} &\equiv \frac{3H^2}{8\pi G} \\
&= 1.9 \times 10^{-29} h^2 \text{ g cm}^{-3} \\
&= 2.8 \times 10^{11} h^2 \, M_\odot \text{ Mpc}^{-3} \\
&= 1.1 \times 10^{-5} h^2 \text{ protons cm}^{-3}
\end{aligned} \tag{25}
$$

Now we can define all densities relative to the critical density and work with dimensionless parameters.

$$
\Omega_{i,0} \equiv \frac{\rho_{i,0}}{\rho_{c,0}}, \quad \text{with} \quad i = radiation, matter, \Lambda \tag{26}
$$

Now writing Eq. (24) in terms of the critical density we get,

$$
\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda \tag{27}
$$

where Ω_Λ is the curvature density parameter, $\Omega_\Lambda \equiv \frac{-kc^2}{H_0^2}$. Now solving both sides at the present time with $a(t_0) \equiv 1$, leads to the constraint

$$
1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda \quad \text{where} \quad \Omega_0 = \Omega_r + \Omega_m + \Omega_\Lambda \tag{28}
$$

Now we consider a flat universe ($k = 0$) with a single fluid component. The different dependence of radiation, matter, and vacuum energy show that individual components dominate different eras of the universe. Therefore parameterising this component in the form of w_i and then updating the Eq. (27) by reducing it to the following form:

$$
\frac{dln\ a}{dt} \approx H_0 \sqrt{\Omega_i} a^{-\frac{3}{2}\ (1+w_i)} \tag{29}
$$

Integrating the equation, we can deduce the time dependence of the scale factor

$$a(t) \propto \begin{cases} t^{2/3(1+w)} & w \neq -1\,, \\ e^{Ht} & w = -1\,, \end{cases} \tag{30}$$

Keeping in mind the assumption of a flat universe, the conformal time τ is defined as

$$d_h(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)} \tag{31}$$

we write the solutions of FLRW spacetime for the matter, radiation and cosmological constant dominated times as shown in the given table:

Table 2. Solutions to the Friedmann Equations

	w	$\rho \sim a^{-3(1+w)}$	$a(t) \sim t^{\frac{2}{3(1+w)}}$	$a(\tau) \sim \tau^{\frac{2}{(1+3w)}}$
Matter Dominated	0	a^{-3}	$t^{\frac{2}{3}}$	τ
Radiation Dominated	$\frac{1}{3}$	a^{-4}	$t^{\frac{1}{2}}$	τ^2
Vacuum Energy Dominated	-1	a^0	e^{Ht}	$-\tau^{-1}$

By using these results and placing them in Eq. (11), we derive that for the standard model of big bang cosmology $\ddot{a} < 0$. This means that the universe is undergoing decelerated expansion which brings us to the problems of the Standard Model of the Big Bang Cosmology.

PROBLEMS WITH THE STANDARD MODEL OF BIG BANG COSMOLOGY

In this section, we will discuss about the various problems of the standard model of big bang cosmology.

Flatness Problem

To understand the flatness problem, first we will write Eq. (24) in the following way,

$$\Omega - 1 = \frac{K}{a^2H^2} \tag{32}$$

As described by the standard big bang model with $\ddot{a} < 0$, we can see that the a^2H^2 term in

Eq. (32) is always decreasing. This can be understood by taking the time derivative of the aH term and then finding it's proportionality to the particles in the universe using the second Friedmann Eq. (11).

$$\frac{1}{aH} \propto (1+3w) \tag{33}$$

We can infer that for the standard cosmology, the comoving Hubble radius, $(aH)^{-1}$, grows with time and from Eq. (33) the quantity $|\Omega - 1|$ therefore must diverge with time. Therefore we understand that the critical value $\Omega = 1$ is an unstable fixed point. However present experimental evidence shows that the value of Ω currently is well within $\mathcal{O}(1)$. Therefore Ω must be very close to the order of unity in the early universe. This requires an extreme fine tuning of the initial conditions because if the values is greater or lesser, the universe either collapses very soon after forming or just expands very fast such that structure formation does not take place. This problem is therefore dubbed the flatness problem.

Horizon Problem

Reiterating that we are working in comoving coordinates from Eq. (31), we can use the fact that the null geodesics are straight lines and the distance between two points is equal to the corresponding difference in conformal time $\Delta\tau$. We can then take a comoving wavelength λ and a physical wavelength $a\lambda$ within the Hubble radius H^{-1}. In standard big-bang cosmology with $0 < p < 1$, the physical wavelength grows as $a\lambda \propto t^p$, while the Hubble radius evolves as $H^{-1} \propto t$. Consequently, the physical wavelength becomes much smaller than the Hubble radius over time, confining causality to a small fraction of the Hubble radius. If we assume that the Big Bang started with the singularity at $t_i \equiv 0$, then the (comoving) particle horizon is defined as the greatest comoving distance from which an observer at time t is able to receive signals travelling at light speed. Or in other words we can also define it as the maximal distance light can travel from $t = 0$ to a time t. The particle horizon $D_H(t)$ then can be defined as:

$$D_H(t) = a(t)d_h(\tau) \tag{34}$$

The Hubble radius is defined as the distance by particles travelled in the course of one expansion time. Therefore the comoving Hubble radius is defined as $(aH)^{-1}$. Now to get more clarity on the arising problem, let's take an example ratio of the particle horizon of CMB at the time of decoupling and today.

$$\frac{d_H(t_{\text{dec}})}{d_H(t_0)} \approx \left(\frac{t_0}{t_{\text{dec}}}\right)^{1/3} \approx \left(\frac{10^5}{10^{10}}\right)^{1/3} \approx 10^{-2} \tag{35}$$

This shows that the causally connected patches only turn out to be a small proportion of the Hubble Radius and that in standard cosmology, the CMB consists of over 40,000 causally disconnected patches of space. Considering that there wasn't enough time for these regions to communicate, there must be some underlying theory/phenomenon which can't be explained by the standard model of cosmology and is therefore called the horizon problem.

Other Problems

Other problems arising from the Standard Model of Big Bang Cosmology, include the Monopole problem and the Large Scale Structure problem. These problems arise from the point of super symmetry breaking which deal with the more fundamental problems of particle physics and therefore lead to the production of artefacts in the form of monopoles, cosmic strings, and other topological defects.

These problems deal with more fundamental issues of particle physics, which arise at the point of super symmetry breaking which leads to the production of relics such as monopoles, cosmic strings and other topological defects. Standard cosmology fails to explain origin of large-scale structure in almost the same way as the Horizon Problem. The last scattering surface has anisotropies with very small amplitudes that are very small and are almost scale-invariant which again cannot be explained using the standard big bang cosmology model.

INFLATION

The idea of inflation was first developed independently in 1981 by Alan Guth [79] and Katsuhiko Sato [157]. This model of inflation is now commonly known as old inflation which talks about de-Sitter inflation where we have first order transition to vacuum. After this in 1982, a new model of inflation was given by Linde [124], and Albrecht and Steinhardt [8] which used second order transition to true vacuum which is the most common model of inflation called slow roll inflation. Since then in the coming fourty years till 2024 there have been several inflationary models in GR and Non GR based theories [1, 56, 73, 94, 114, 117–120, 163, 172, 173, 175].

From the above descriptions of the given problems, we can see why the concept of comoving Hubble radius $(aH)^{-1}$ is very important to the fundamental problems of the Standard Big Bang Cosmology Model such as the horizon problem and the flatness problem. Both of these problems arise because the comoving Hubble radius is strictly increasing. The idea of inflation then uses the proposition that at the start of the universe there was a period with a shrinking comoving Hubble radius. This ensures that the comoving Hubble radius decreases sufficiently in the very early universe and therefore the problems with the standard model of cosmology can be solved.

Inflation As a Solution To The Problems of Standard Big Bang Model

The idea of a shrinking comoving Hubble radius is possible when the strong energy conditions are violated, as shown below:

$$1 + 3w < 0 \tag{36}$$

Flatness Problem Solved

Due to the shrinking comoving Hubble radius the value of Ω from the Eq. (33) actually converges to 1 instead of being an unstable fixed point like in the case of the standard model of big bang cosmology. Another way to look at the solution is to work out the evolution of the curvature parameter in terms of the evolution of the comoving Hubble radius which can be given as

$$\Omega_k(N) = \frac{(a_i H_i)^2}{(aH)^2}\Omega_k(t_i) \tag{37}$$

As it is clear from the equation, the initial curvature will decrease if the comoving Hubble radius decreases. Therefore if inflation lasts long enough it solves the flatness problem.

Horizon Problem Solved

Now we employ the shrinking comoving Hubble radius to the horizon problem, by implying the following relation:

$$\frac{d}{dt}(aH)^{-1} < 0 \tag{38}$$

From here $aH = \dot{a}$, and therefore we can clearly see that $\ddot{a} > 0$, therefore calculating the dependence, we derive an exponentially expanding phase of the early universe.

To explicitly show this we can write Eq. (31) in the following way

$$d_h(\tau) = \int_{t_i}^{t} \frac{dt}{a(t)} = \int_{a_i}^{a} \frac{da}{a\dot{a}} = \int_{lna_i}^{lna} (aH)^{-1} dlna \tag{39}$$

If the period of inflation lasts long enough we can see that all the physical scales that have left the Hubble Radius during the radiation or matter dominated phase can re-enter it during the past.

We see that the physical Hubble rate is constant during inflation, therefore the amount by which the comoving Hubble radius decreases is equal to the amount by which the scale factor increases. The amount by which the scale factor increases is called the e-folding number and can be shown as

$$N_{tot} \equiv ln(a_e/a_i) \tag{40}$$

The reheating temperature (T_R) shows the dependence of the amount of increase of the

Hubble radius with respect to the greatest temperature associated with the thermal plasma at the start of the hot big bang. Now we know that during the radiation dominated era the Hubble constant falls off as a^{-2}, we have

$$\frac{a_0 H_0}{a_R H_R} = \frac{a_0}{a_R}\left(\frac{a_R}{a_0}\right)^2 = \frac{a_R}{a_0} \sim \frac{T_0}{T_R} \sim 10^{-28}\left(\frac{10^{15} GeV}{T_R}\right) \tag{41}$$

Where we used a reference value of $10^{15} GeV$ signifying the reheating temperature. We also undergo the assumption that when inflation ends the energy density is quickly converted to the particles of thermal plasma, therefore the Hubble radius didn't significantly grow during the reheating period which we have defined at the end of inflation and at the beginning of the Hot Big Bang. This can be written as:

$$a_i {H_i}^{-1} > a_0 {H_0}^{-1} \sim 10^{-28}\left(\frac{10^{15} GeV}{T_R}\right) \tag{42}$$

Using $H_i = H_e$, we get

$$N_{tot} \equiv ln(a_e/a_i) > 64 + ln\left(\frac{T_R}{10^{15} GeV}\right) \tag{43}$$

This gives us the solution that at least 60 e-folds are required during inflation.

Solutions To The Other Problems

Inflation also gives an appropriate solution to the Large Scale Structure problem. The understanding that the comoving Hubble radius during inflation decreases, generates nearly scale invariant density perturbations on large scales. During the initial phase of inflation, as perturbations exist within the Hubble radius, the workings of causal physics lead to the generation of small quantum fluctuations. Once a scale exits the Hubble radius(i.e., undergoes the first horizon crossing) during inflation, the perturbations can be characterized as classical. Subsequently, as the inflationary period concludes, the universe's evolution transitions to the standard big-bang cosmology, initiating an increase in the comoving Hubble radius. Subsequently, the scales of perturbations re-enter the Hubble radius (the second horizon crossing), marking the resumption of causality. Following the second horizon crossing, the small perturbations imprinted during inflation manifest as large-scale perturbations. These classical density perturbations then explain the mechanism of the CMB anisotropies and the Large Scale Structure formation. This is further shown in Fig. 3.

For the solution to the Monopole problem, we consider that the physical Hubble radius remains constant during inflation therefore the energy density of the universe decreases very slowly with respect to time, whereas the energy density of the massive particles falls off much faster ($\sim a^{-3}$), due to which these particles undergo redshift during inflation which naturally works as a solution to the monopole problem.

INFLATIONARY DYNAMICS

In this section, we want to derive the conditions necessary for inflation to occur given the above description along with the e-folding number. These basic conditions without saying anything about the mechanism driving the exponential expansion are called the Hubble slow roll parameters. As we saw earlier the first condition of inflation is that all the physical quantities are slowly varying. Rewriting the comoving Hubble radius in the following form we get

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} \tag{44}$$

This can be further evaluated as

$$\frac{\ddot{a}}{a} = H^2(1-\varepsilon), \qquad \text{where} \quad \varepsilon \equiv -\frac{\dot{H}}{H^2} \tag{45}$$

where ε is the first slow roll parameter, which is mathematically given as:

$$\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN} < 1 \tag{46}$$

using the fact that $dN \equiv dlna = Hdt$. Therefore it is observed that for a shrinking comoving

Hubble radius, we have $\varepsilon < 1$. Therefore we require the first slow roll parameter to be less than one. Now we also want that inflation lasts for a sufficient long time(about 60 e-folds as we calculated in the last section). Now we have to make sure that ε remains small for a sufficiently large number of Hubble times. We can quantify these conditions using a second slow roll parameter.

$$\kappa \equiv \frac{dln\varepsilon}{dN} = \frac{\dot{\varepsilon}}{H\varepsilon} \tag{47}$$

Now we need $|\kappa| < 1$, these parameters are called Hubble slow roll parameters and are required for inflation to occur.

Dynamics of Scalar Fields

In the earlier section, we have seen how inflation is a pertinent solution for many of the problems of the standard model of big bang cosmology. In this section, we will now want to quantify what are the mechanisms we can imply to have a period of inflation. During the era of inflation, the universe can be assumed to be dominated by a scalar field. Lets assume a homogeneous scalar field $\phi(t,x)$, also known as the inflaton where the value of the field can depend on time t and position x. The potential of this scalar field can be assumed to be driving the exponential expansion of the universe. The simplest models of this type are minimally coupled scalar field models. The Lagrangian for the same can be written as:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - V(\phi)\,, \tag{48}$$

Here we can define $V(\phi)$ according to the model of our choice. If the scalar fields dominates the universe then we can take it as the source of the FLRW background. First we take a scalar field in expanding FLRW spacetime which we can define by taking the action as:

$$S = \int d^4x\, a^3(t)\left[\frac{1}{2}\dot{\phi}^2 - \frac{(\nabla\phi)^2}{2a^2(t)} - V(\phi)\right] \tag{49}$$

Now we apply the evolution of the homogeneous field configuration such as $\phi = \phi(t)$, we can then reduce the action to the following form as the gradient term disappears

$$S = \int d^4x\, a^3(t) \left[\frac{1}{2}\dot{\phi}^2 - \frac{(\nabla\phi)^2}{2a^2(t)} - V(\phi) \right] \tag{50}$$

Now we take the variation in the scalar field to be such that $\phi \longrightarrow \phi + \delta\phi$, we can write the change in the action as

$$\delta S = S = \int d^4x\, a^3(t) \left[\dot{\phi}\delta\dot{\phi} - \frac{dV}{d\phi}\delta\phi \right] \tag{51}$$

Now taking $\delta\phi$ out from the equation we get,

$$\delta S = \int d^4x\, \left[-\frac{d}{dt}(a^3\dot{\phi}) - a^3\frac{dV}{d\phi} \right] \delta\phi \tag{52}$$

Now applying the principle of least action we can use the fact that $\delta S = 0$ which leads us to the Klein-Gordon Equation

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \tag{53}$$

Here $3H\dot{\phi}$ is called the friction term. This friction term is necessary for the potential to be flat and one more interesting fact is that large field inflation models are more viable due to this friction term, instead of small field inflation models. Now assuming that this scalar field dominates the universe, to use the Friedmann equations, we will now derive the energy and pressure associated with the field. Given the action in Eq. (50), we can observe that the energy density is the sum of kinetic and potential energy densities.

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{54}$$

Lets take the time derivative of the energy density, we get

$$\dot{\rho}_\phi = \left(\ddot{\phi} + \frac{dV}{d\phi} \right) \dot{\phi} = -3H\dot{\phi}^2 \tag{55}$$

where we have used the Klein Gordon equation to get the equality. If we compare this to the continuity equation (21), we can formulate the pressure induced by the field:

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{56}$$

Next we will discuss the case for slow roll inflation.

Slow-roll Inflation

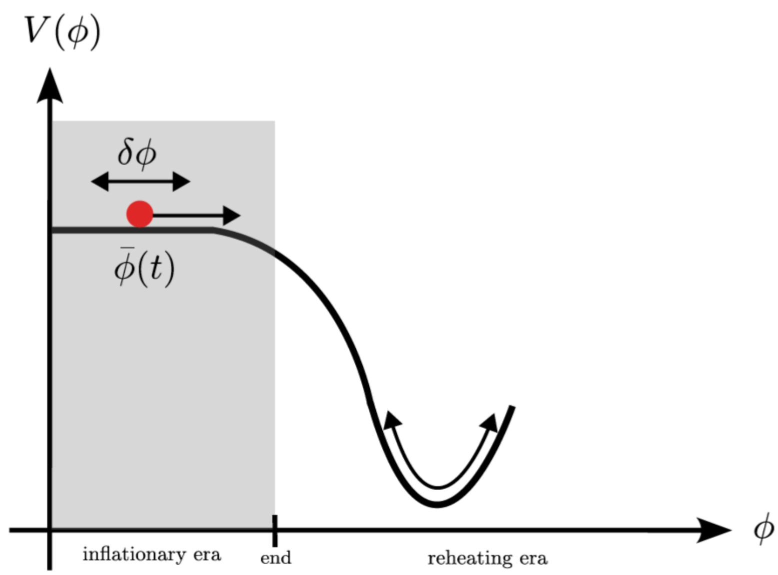

Figure 1. Slow Roll Potential Curve

First we will write the Friedmann equation in the terms of the Hubble Parameter and the energy density

$$H^2 = \frac{1}{3{M_{Pl}}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right] \tag{57}$$

where we have defined the Planck mass in the following way, $M_{Pl} = \sqrt{\frac{\hbar c}{G}}$. Coupling the Friedmann and the Klein Gordon Eq. (53), we can get

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{{M_{Pl}}^2} \tag{58}$$

Taking the ratio of Eq. (57) and Eq. (58), we derive the first slow roll parameter

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{{M_{Pl}}^2 H^2} = \frac{\frac{3}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V} \tag{59}$$

Now for inflation to be sustained we need $\varepsilon \ll 1$, therefore we need the kinetic term $\frac{1}{2}\dot{\phi}^2$ to make a very small contribution with regards to the total energy density, $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V$. Therefore due to relatively less contribution of the kinetic term the model is appropriately called slow roll inflation.

For the slow roll behaviour to persist, the acceleration of the scalar field has to be small or in other words we want the potential to be sufficiently flat. Therefore we define the dimensionless acceleration per Hubble time as:

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{60}$$

Taking the time derivative of the first slow roll parameter we have,

$$\dot{\varepsilon} = \frac{\dot{\phi}\ddot{\phi}}{{M_{Pl}}^2 H^2} - \frac{\dot{\phi}^2 \dot{H}}{{M_{Pl}}^2 H^3} \tag{61}$$

Substituting this into Eq. (47), we get

$$\kappa = \frac{\dot{\varepsilon}}{H\varepsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} = 2(\varepsilon - \delta) \tag{62}$$

Now we see that $\varepsilon, |\delta| \ll 1 \implies \varepsilon, |\kappa| \ll 1$. Hence we see if both the speed and the acceleration of the inflaton are small, then we can say that the inflation will last for a long time. Now we will use the derived conditions for the slow roll parameters to derive the reduced and simpler equations due to the homogenity and isotropy of the background.

$$H^2 \approx \frac{V}{3{M_{Pl}}^2} \tag{63}$$

Therefore the Hubble expansion rate is only determined by the potential and from the $|\delta| \ll 1$, only the friction term dominates and the Klein-Gordon reduces to the form

$$3H\dot{\phi} \approx -V_{,\phi} \quad \text{where} \quad V_{,\phi} = \frac{dV}{d\phi}. \tag{64}$$

Now calculating the first slow roll parameter in these conditions which gives the relation between the slope of the potential and the speed of the inflaton. We get

$$\varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{{M_{Pl}}^2 H^2} = \frac{{M_{Pl}}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \tag{65}$$

Here we have the first slow roll parameter completely in terms of the potential. Now taking the time derivative of the reduced Klein-Gordon equation we get

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = V_{,\phi\phi}\dot{\phi} \tag{66}$$

which leads to the second potential slow roll parameter

$$\delta + \varepsilon = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} \approx {M_{Pl}}^2 \frac{V_{,\phi\phi}}{V} \tag{67}$$

Therefore a robust way of testing the potential leading to slow roll inflation is to calculate the potential slow roll parameters

$$\varepsilon_{\mathrm{v}} = \frac{{M_{Pl}}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \qquad \eta_{\mathrm{v}} = {M_{Pl}}^2 \frac{V_{,\phi\phi}}{V} \tag{68}$$

We have successful inflation if both of these parameters are much smaller than 1. Inflation is further explained in great detail in these books [29,31,59,115,119]. Some more general concepts for gravitation and cosmology are covered in the following books [44, 59, 137, 178].

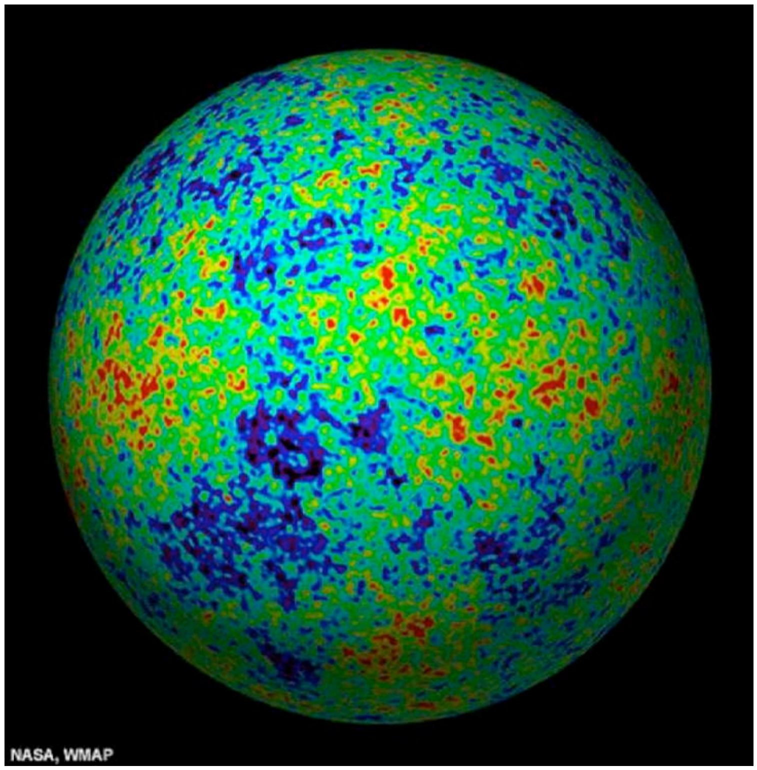

Figure 2. CMB radiation sphere projection

In the earlier section we saw how the theory of inflation solves the problems of the standard model of big bang cosmology. Now to verify it we will fit the theoretical formulation with the experimental evidence in the form of the CMB observed through the COBE, WMAP and Planck Telescopes. Even though we assume the cosmological principle for the FLRW background, we observe that the universe is not perfectly homogeneous through the CMB map and has anisotropies of the lower order of $\sim 10^{-5}$ [60, 72, 116, 125, 126]. Inflation accounts for these anisotropies through the introduction of the quantum fluctuations on sub horizon scales at the start of the inflationary period. To characterise these fluctations, the theory of cosmological perturbation theory was introduced by Bardeen in 1980 [16]. Further developments in this theory were given by Kodama and Sasaki in their seminal work in 1984 [99]. Further significant advances in the work were done by Mukhanov in 1992 [142]. A modern theory was given by Malik and Wands in 2009 [132]. Now we develop the observable quantities in the cosmological perturbation theory.

During the inflationary epoch, as the Universe undergoes rapid expansion, the comoving Hubble radius decreases over time, eventually becoming smaller than the comoving wavelength of fluctuations. As a result, when these fluctuations exit the horizon, they become causally disconnected and remain in a frozen state until the end of inflation. Subsequently, they gradually re-enter the horizon as classical density perturbations. During inflation the inflaton's energy dominates, the field equations through the spacetime geometry are affected through the perturbations of the inflationary field. As the metric is symmetric under SO(3) in the background spacetime the perturbations can be decoupled at first order into total 10 independent degrees of freedom can be divided into 4 scalar modes, 4 vector modes, and 2 tensor modes.

Newtonian Perturbation Theory

We assume the Newtonian gravity approximation for the case when the velocities are small compared to the speed of light and we assume an almost flat universe. First we will build intuition for the simplified case and then move to the FLRW case. Assuming linear perturbations for the background metric, we see how small perturbations around the solution in density and pressure evolve in time. We first modify the continuity Eq. (15) and Euler Eq. (16) by taking a fluid with mass density ρ, pressure $P \ll \rho$ and velocity u.

$$\begin{aligned} \rho(t,x) &= \bar{\rho} + \delta\rho(t,x) \\ P(t,x) &= \bar{P} + \delta P(t,x) \end{aligned} \tag{69}$$

Now we apply the perturbed solutions to the continuity and the Euler equations. And then we first get the perturbed continuity equation as

$$\partial_t \delta\rho = -\nabla.(\bar{\rho} u) \tag{70}$$

And then we can also get the perturbed Euler equation as

$$\bar{\rho}\partial u = -\nabla \delta P \tag{71}$$

Combining ∂_t(70) and ∇(71) gives:

$$\partial_t^2 \delta\rho - \nabla^2 \delta P = 0 \tag{72}$$

Now we write this purely in terms of the density fluctuations $\delta\rho$, using the relation between P and ρ provided by the equation of state. Again considering the barotropic fluid assumption, the pressure perturbation becomes of the form

$$\delta\rho = \frac{\partial P}{\partial \rho}\delta\rho \equiv c_s^2 \delta\rho \tag{73}$$

where c_s^2, is the sound speed of the fluid. we can then write Eq. (72) as

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2\right)\delta\rho = 0 \tag{74}$$

Which gives us a wave equation and the solutions are in the form of sound waves given as:

$$\delta\rho(x,t) = A_k\, sin(\omega\, t - k\cdot x) + B_k\, cos(\omega\, t - k\cdot x) \tag{75}$$

where we have $\mathbf{k}$ as the wavevector and $\omega = c_s|k|$ is the frequency. Since the equation is linear we can write solutions with different wavevectors k which can be solved more formally by switching into fourier space as it turns the partial differential equation into an ordinary differential equation. We write the fourier transform of $\delta\rho$ in the following way:

$$\delta\rho(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \delta\rho_(\mathbf{k},t). \tag{76}$$

When this is acted upon a spatial derivative ∂_i pulls down a factor of ik_i from $e^{i\mathbf{k}\cdot\mathbf{x}}$ and the Laplacian in (74) becomes $-k^2$, with $\mathbf{k} = |\mathbf{k}|$, in Fourier space. Therefore we then get the ordinary differential equation as:

$$\left(\frac{\partial^2}{\partial t^2} + c_s^2 k^2\right)\delta\rho_(\mathbf{k},t) = 0. \tag{77}$$

The solution for each Fourier mode then is

$$\delta\rho_{\mathbf{k}}(t) = C_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + D_{\mathbf{k}} e^{i\omega_{\mathbf{k}} t}, \tag{78}$$

with $\omega_{\mathbf{k}} = c_s k$. Substituting this back into (76) gives

$$\delta\rho(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} (C_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + D_{\mathbf{k}} e^{i\omega_{\mathbf{k}} t}) \tag{79}$$

$$= \int \frac{d^3k}{(2\pi)^3} (C_{\mathbf{k}} e^{-i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})} + D_{-\mathbf{k}} e^{i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})}) \tag{80}$$

$$= \int \frac{d^3k}{(2\pi)^3} (C_{\mathbf{k}} e^{-i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})} + C^*_{\mathbf{k}} e^{i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})}), \tag{81}$$

In the second line's second term, we replace $\mathbf{k}$ with $-\mathbf{k}$ and establish that $D_{-\mathbf{k}}$ equals the complex conjugate of $C_{\mathbf{k}}$ in the concluding line. This step is essential to ensure that the density fluctuations $\delta\rho(\mathbf{x},t)$ remain real. Consequently, it becomes evident that the most comprehensive solution for the wave equation is, in fact a sum (or integral) over the plane wave solutions in (75). Now we will add complexities such as gravity and expansion of the universe to the formalism to take it to the FLRW background. Firstly the Newtonian Gravity under these assumptions is given by the Poisson equation as follows,

$$\nabla^2\Phi = 4\pi G\rho \tag{82}$$

And then we account for the expansion of the universe by writing the physical coordinates r in terms of the comoving coordinates $\mathbf{x}$,

$$\mathbf{r}(t) = a(t)\,\mathbf{x} \tag{83}$$

Therefore now we can introduce perturbations about the background solution

$$\rho = \bar{\rho}(1+\delta), \quad \mathbf{u} = H\mathbf{r} + \mathbf{v}, \quad P = \bar{P} + \delta P, \quad \phi = \bar{\phi} + \delta\phi, \tag{84}$$

where we have defined the density contrast as:

$$\delta = \frac{\delta\rho}{\bar{\rho}} \tag{85}$$

Now with the following formalism we can expand the equations of motion to linear order perturbation theory. The updated equations are given as follows

$$\text{Continuity equation:} \quad \dot{\delta} = -\frac{1}{a}\nabla\cdot\mathbf{v}, \tag{86}$$

$$\text{Euler equation:} \quad \mathbf{v}+H\mathbf{v} = -\frac{1}{a}\nabla\delta P - \frac{1}{a}\nabla\delta\phi, \tag{87}$$

$$\text{Poisson equation:} \quad \nabla^2\delta\phi = 4\pi G a^2\bar{\rho}\delta, \tag{88}$$

where now ∇ is a derivative with respect to $\mathbf{x}$ and the dot represents a time derivative at a fixed $\mathbf{x}$. We can then combine all the linear equations to form a combined equation for the evolution of the density contrast which is given as:

$$\ddot{\delta} + 2H\dot{\delta} - \left(\frac{c_s^2}{a^2}\nabla^2 + 4\pi G\rho(t)\right)\delta = 0, \tag{89}$$

Taking into account that $\delta P = c_s^2\delta\rho$ for a barotropic fluid, we observe that on smaller scales the pressure term is dominant, leading to oscillations determined by the speed of sound, denoted as c_s. In contrast, on broader scales, the significance of pressure diminishes and gravitational effects take precedence. This pattern of fluctuation growth is known as Jeans instability. This concept has been applied to the development of dark matter perturbations. In the era of radiation dominance, the swift expansion of the cosmos works against gravitational instability, resulting in a growth of perturbations that can be described as logarithmic, specifically $\delta \sim \ln a$. When matter takes over as the dominant constituent of the universe, the growth pattern of the perturbations shifts to a linear scale with $\delta \sim a$.

Relativistic Perturbation Theory

Now we will relax the Newtonian approximation and work in full GR. Since our background metric is FLRW, perturbations for the FLRW background can be written in terms of metric $\bar{g}_{\mu\nu}$ and energy momentum tensor $\bar{T}_{\mu\nu}$ perturbations, which can be defined in the following way

$$T_{\mu\nu}(t,\mathbf{x}) = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}(t,\mathbf{x}), \qquad g_{\mu\nu}(t,\mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\mathbf{x}) \tag{90}$$

where we can then define the perturbed metric as,

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -(1+2\Phi)\mathrm{d}t^2 + 2aB_i\mathrm{d}x^i\mathrm{d}t + a^2\left[(1-2\Psi)\delta_{ij} + E_{ij}\right]\mathrm{d}x^i\mathrm{d}x^j, \end{aligned} \tag{91}$$

where we have defined scalar Φ as lapse, vector B_i as shift, scalar Ψ as the spatial curvature perturbation, and E_{ij} is a spatial shear 3-tensor which is symmetric and traceless, $E^i_i = \delta^{ij}E_{ij} = 0$. We define the 3-surfaces of constant time t as slices and curves of constant spatial coordinates x^i. On the other hand varying time t are called threads. In real space, the SVT decomposition which corresponds to the distinct transformation properties of scalars, vectors and tensors on spatial hypersurfaces of the metric perturbations Eq. (90) is given as:

$$B_i \equiv \partial_i B - S_i, \qquad \text{where} \quad \partial^i S_i = 0, \tag{92}$$

and

$$E_{ij} \equiv 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}, \qquad \text{where} \quad \partial^i F_i = 0, \quad h^i_i = \partial^i h_{ij} = 0. \tag{93}$$

Inflation is not responsible for the creation of the vector perturbations S_i and F_i, which isn't of much concern as the vectors usually come along with gradients which decompose quickly during inflation. Therefore we will only study scalar and tensor fluctuations which are then observed as density fluctuations(leading to Large Scale Structure) and gravitational waves in the late universe.

Tensor fluctuations are gauge-invariant, but scalar fluctuations change under a change of coordinates. Therefore the scalar metric perturbations are not uniquely defined and depend of the choice of coordinates or gauges. While writing down the perturbed metric Eq. (91), we implicitly introduced a specific time slicing and defined specific spatial coordinates on these time slices. Changing the coordinates can also change the values of the perturbation variables. It may also lead to the problem of fictitious perturbations or gauge modes which are fake perturbations arising from an incorrect choice of coordinates even when the background is perfectly homogeneous. Now we will consider the following coordinate transformations,

$$t \rightarrow t+\alpha \tag{94}$$

$$x^i \rightarrow x^i+\delta^{ij}\beta_{,j}\,. \tag{95}$$

We can define the scalar metric perturbation transformations under the coordinate transformations as follows:

$$\Phi \rightarrow \Phi-\dot{\alpha} \tag{96}$$

$$B \rightarrow B+a^{-1}\alpha-a\dot{\beta} \tag{97}$$

$$E \rightarrow E-\beta \tag{98}$$

$$\Psi \rightarrow \Psi+H\alpha\,. \tag{99}$$

We will use gauge invariant variables to tackle the issue of fictitious gauge modes described above. We can define the first gauge invariant scalar quantity in the form of the curvature perturbation on uniform-density-hypersurfaces.

$$-\zeta \equiv \Psi+\frac{H}{\dot{\rho}}\delta\rho\,. \tag{100}$$

where ρ is the total energy density of the universe. Geometrically, ζ measures the spatial curvature of constant-density hypersurfaces, $R^{(3)} = 4\nabla^2\Psi/a^2$.

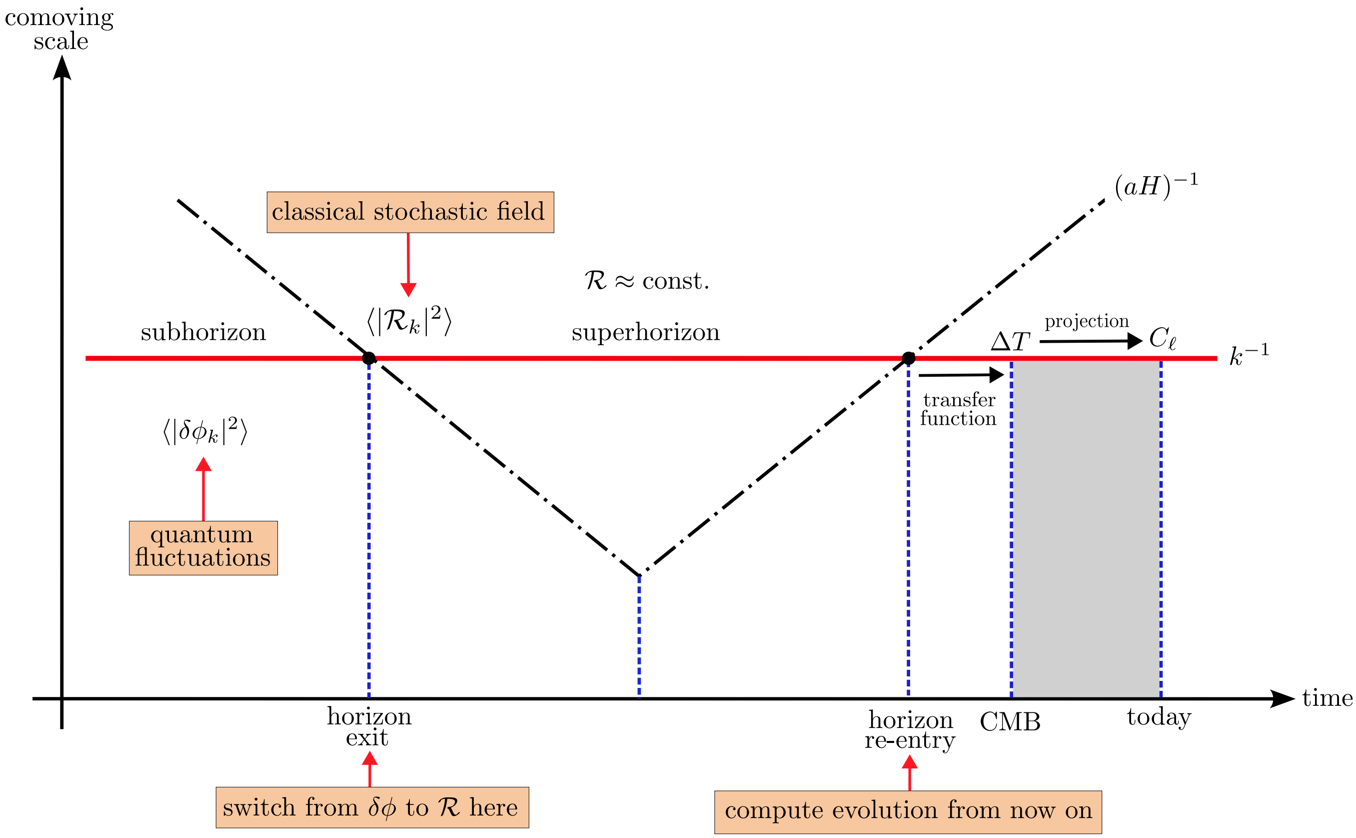

Figure 3. Evolution of perturbations with the Comoving Horizon

We define the horizon crossing at $k = aH$ where k is the comoving wavelength of a mode. Subsequently we can define the sub horizon $k < aH$ and the super horizon where we can also write $k << aH$ during inflation in it's vacuum stage. A classical probability density, which is determined by the power spectrum is used to define a mode which exists the horizon during contraction given by $c_s k = aH$ where c_s is again the sound speed. Since we are working with the FLRW background the sound speed just becomes the speed of light. As the fluctuations freeze out after horizon crossing $k \ll aH$, ζ remains constant outside the horizon for adiabatic matter perturbations, which is given by the following equation

$$\delta p_{ad} \equiv \delta p - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} \delta \rho \tag{101}$$

δp_{ad} here is also gauge-invariant. There are also isocurvature perturbations which can arise after the first horizon crossing but they are not experimentally verifiable and are therefore often not considered. The condition (101) is satisfied for single field inflation models, as long as the perturbation $\zeta_{\mathbf{k}}$ doesn't evolve outside the horizon, $k \ll aH$. For the spatially flat gauge or the comoving gauge defined by spatially flat hypersurfaces, the spatial curvature of the universe is set to zero, and the spatial coordinates are chosen such that they expand with the universe, making them comoving with the cosmic expansion. For this gauge Ψ, the perturbations ζ is the dimensionless density perturbation $\frac{1}{3}\delta\rho/(\bar{\rho}+\bar{p})$. More information of different gauge choices is given in Appendix. A. Now we will define a very important quantity known as the transfer function which defines the k-dependent growth of the fluctuations during the horizon entry $k = aH$ given as:

$$T(k) \equiv \frac{D_+(t_i)\delta(k,t)}{D_+(t)\delta(k,t_i)} \tag{102}$$

On sub-horizon scales ζ is related to density perturbations $\delta\rho$. Taking into account appropriate transfer functions to describe the sub-horizon evolution of the fluctuations, we can then use to relate observables of the CMB and LSS to the primordial value of ζ, which during slow roll inflation is given as:

$$-\zeta \approx \Psi + \frac{H}{\dot{\bar{\phi}}} \delta\phi \,. \tag{103}$$

We can also define the comoving curvature perturbation which is also a gauge-invariant scalar as:

$$\mathcal{R} \equiv \Psi - \frac{H}{\bar{\rho}+\bar{p}} \delta q \,, \tag{104}$$

where δq is the scalar part of the 3-momentum density $T_i^0 = \partial_i \delta q$. During inflation $T_i^0 = -\dot{\bar{\phi}}\, \partial_i \delta\phi$ and therefore we can then rewrite it as:

$$\mathcal{R} = \Psi + \frac{H}{\dot{\bar{\phi}}}\delta\phi \tag{105}$$

Geometrically, we see that the spatial curvature of comoving (or constant-ϕ) hypersurfaces is given by $\mathcal{R}$. We can now use the linearized Einstein equations to relate ζ and $\mathcal{R}$ as follows:

$$-\zeta = \mathcal{R} + \frac{k^2}{(aH)^2}\frac{2\bar{\rho}}{3(\bar{\rho}+\bar{p})}\Psi_{\rm B}\,, \tag{106}$$

where

$$\Psi_{\rm B} \equiv \psi + a^2 H(\dot{E} - B/a)\,, \tag{107}$$

is a Bardeen potential [16]. ζ and $\mathcal{R}$ are therefore equal on superhorizon scales, $k \ll aH$. ζ and $\mathcal{R}$ are also equal during slow-roll inflation, given by equations (103) and (105). The correlation functions of ζ and $\mathcal{R}$ are therefore equal at horizon crossing and both ζ and $\mathcal{R}$ are conserved on superhorizon scales.

The Einstein equations then give the evolution equation for the gauge-invariant curvature perturbation

$$\dot{\mathcal{R}} = -\frac{H}{\bar{\rho}+\bar{p}}\,\delta p_{en} + \frac{k^2}{(aH)^2}\Big(\dots\Big)\,. \tag{108}$$

Adiabatic matter perturbations satisfy $\delta p_{ad} = 0$ and $\mathcal{R}$ is conserved on superhorizon scales, $k < aH$.

Statistical Properties

In this section, we will now define and derive the statistical properties of these perturbations and link them to observables. The power spectrum of $\mathcal{R}$ (or ζ)(as both are the same after first horizon crossing) for the primordial scalar fluctuations is defined through the two point correlation functio. The two point correlation function describes how field values at different points are related to each other. The power spectrum arising from this correlation function quantifies how fluctuations are distributed across different scales(wavenumbers). The two point correlation function for the primordial comoving curvature perturbations is given as:

$$\langle \zeta_{\mathbf{k}_1}\ \zeta_{\mathbf{k}_2} \rangle = (2\pi)^3\, \delta^3(k_1 + k_2)\, P_\zeta(k_1)\,, \qquad \mathcal{P}_\mathcal{S} = \frac{k^3}{2\pi^2} P_\zeta(k)\,. \tag{109}$$

Here, $\langle \ldots \rangle$ defines the ensemble average of the fluctuations. We see that the result is proportional to the Dirac delta function which is a consequence of translation invariance. From this delta function we can see the implication that Fourier modes with different wave vectors are independent of each other. The power spectrum is often approximated by a power law form

$$\mathcal{P}_\mathcal{S} = \frac{k^3}{2\pi^2} P_\zeta(k) = A_s(k_{\rm ref}) \left(\frac{k}{k_{\rm ref}} \right)^{n_s - 1} = \tilde{A}_s k^{\tilde{n}_s - 1}, \tag{110}$$

where $A_s(k_{\rm ref})(= \tilde{A}_s)$ and n_s are the amplitude and the spectral index at the reference scale $k_{\rm ref}$, respectively. We also introduce $\tilde{k} = k/k_{\rm ref}$ for shorthand notation. Assuming a Gaussian approximation for ζ then the complete statistical information is contained in the power spectrum. The higher order correlation functions of $\mathcal{R}$ give us Primordial Non-Gaussianity. The scalar spectral index determines the scale-dependence of the power spectrum as follows:

$$n_s - 1 = \frac{d \ln \mathcal{P}_\mathcal{S}}{d \ln k}, \tag{111}$$

where scale-invariance corresponds to the value $n_{\rm s} = 1$. The spectral index was first introduced by Harrison, Zel'dovich, and Peebles that the initial perturbations of our universe should take a power law form. Therefore this is now called the Harrison-Zel'dovich spectrum. We can define the running of the spectral index by:

$$\alpha_{\rm s} \equiv \frac{d n_{\rm s}}{d \ln k}\,. \tag{112}$$

In single-field slow-roll inflation the non-Gaussianity is predicted to be small [2, 131], but non-Gaussianity can be significant in multi-field models or in single-field models with non-trivial kinetic terms and/or violation of the slow-roll conditions. We will look at primordial non-Gaussianity in greater detail in the next section. The Planck team has precisely measured these quantities as $A_s = 2.1 \times 10^{-9}$ and $n_s = 0.965 \pm 0.004$ (2σ CL) with $k_{\rm ref} = 0.05\,{\rm Mpc}^{-1}$ in the framework of the standard flat ΛCDM model [3–6, 18, 33], using the Planck temperature + WMAP polarization data.

The power spectrum for the two polarization modes of h_{ij}, *i.e.* $h \equiv h^+, h^\times$, which represents the tensor degrees of freedom is defined as

$$\langle h_{k1}\; h_{k2} \rangle = (2\pi)^3\, \delta^3(k_1 + k_2)\, P_h(k)\,, \qquad \mathcal{P}_\mathcal{H} = \frac{k^3}{2\pi^2} P_h(k)\,. \tag{113}$$

We define the power spectrum of tensor perturbations as the sum of the power spectra for the two polarizations as follows:

$$\mathcal{P}_\mathcal{T} \equiv 2\mathcal{P}_\mathcal{H}\,. \tag{114}$$

Its scale-dependence is defined analogously to Eq. (111) where $n_{\rm t}$ is the tensor spectral index and is defined as follows:

$$n_{\rm t} \equiv \frac{d\ln \mathcal{P}_\mathcal{T}}{d\ln k}\,, \tag{115}$$

i.e.

$$\mathcal{P}_{\rm t}(k) = A_{\rm t}(k_\star) \left(\frac{k}{k_\star}\right)^{n_{\rm t}(k_\star)}\,. \tag{116}$$

By working with the spatially flat gauge and assuming that we are working in a Quasi De-Sitter spacetime, where we have expanded the action of the scalar field to second order using Mukhanov's method and then applied the boundary conditions to the Mukhanov Sasaki equation taking the the Bunch Davies vacuum we can then calculate the scalar and power spectrum at the horizon crossing. We first define the power spectrum for scalar perturbations ($\mathcal{P}_\mathcal{S}$) mathematically given as:

$$\mathcal{P}_{\mathcal{S}} = \left. \frac{H^4}{2k^3\dot{\phi}^2} \right|_{k=aH}. \tag{117}$$

And for tensor perturbations ($\mathcal{P}_{\mathcal{T}}$), we can define it as:

$$\mathcal{P}_{\mathcal{T}} = \left. \frac{4H^2}{k^3 M_{Pl}^2} \right|_{k=aH}. \tag{118}$$

Additionally, the tensor-to-scalar ratio r is defined as:

$$r = \frac{\mathcal{P}_{\mathcal{T}}(k)}{\mathcal{P}_{\mathcal{S}}(k)}. \tag{119}$$

We observe that these quantities are unaffected by a change of scale and are therefore scale invariant quantities. In canonical scalar field models under the assumption of slow-roll approximation, we can express the power spectra only in terms of the field potential $V(\phi)$. Therefore for canonical-scalar field models with slow roll approximation, the spectral indices and the tensor-to-scalar ratio are given as:

$$n_{\mathcal{S}} = 1 - 6\varepsilon + 2\eta, \quad n_{\mathcal{T}} = -2\varepsilon, \quad r = 16\varepsilon. \tag{120}$$

Gravitational Waves are also generated due to scalar and tensor perturbations, some works that describe the process in detail are [12, 25, 32, 169]. The motivation behind the development of cosmological perturbation theory is extensively covered in the following references [27, 40, 41, 111, 132]. Moreover, a deeper examination into cosmological perturbations can be found in the following scholarly works [11, 17, 20, 55, 83, 86, 123, 127, 129, 143, 149, 150, 164]. For a comprehensive background study on cosmological perturbations, the following books are recommended sources [39, 99, 142].

PRIMORDIAL NON GAUSSIANITY

In the previous section we saw how by defining early universe cosmological density perturbations using gaussian random fields is a very good approximation as there is no direct evidence of deviations from the Gaussian hypothesis in the terms of Primordial non-Gaussianity. Still, studying Primordial non-Gaussianity is a very important avenue as it can be used to put constraints on bounds of the early dynamical processes such as inflation which lead to the generation of these initial conditions.

As described in the earlier section, we saw how we can use the statistical correlation functions to derive important parameters regarding to primordial fluctuations such as the two point correlation function which gives us the relation between the curvature perturbation (ζ) and the power spectra. In this section we will now derive the three point and four point correlation functions which give us the bispectrum and trispectrum of ζ.

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle_c = (2\pi)^3 B_\zeta(k_1, k_2, k_3) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \tag{121}$$

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \zeta_{\mathbf{k}_4} \rangle_c = (2\pi)^3 T_\zeta(k_1, k_2, k_3, k_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4), \tag{122}$$

where the subscript c represents the fact that we take the connected part for the correlation functions. Connectedness in this context refers to the parts of the correlation functions that cannot be factored into products of lower-order correlation functions. We take this form as it can be used to isolate the actually interacting components of the field from those correlations that only arise due to statistical fluctuations or the field's Gaussian components.

Bispectrum

The non linearity parameter f_{NL} characterises the amplitude of the bispectrum. As we saw earlier, the power spectrum has only one wave number dependence whereas the bispectrum depends on three wavenumbers. The way the primordial fluctuations are generated dictates the shape and wave number dependence of the bispectrum. Therefore f_{NL} is defined for some specific "shapes" of the bispectrum. We can classify the f_{NL} into four types out of which the two types of interest for the thesis are described below

Local Type

We observe this type of bispectrun when the non-linearities arise in the primordial adiabatic fluctuations at the superhorizon scales where the non-linearities are local in the real space. Here we expand the curvature perturbation(ζ) in the following way:

$$\zeta = \zeta_g + \frac{3}{5} f_{\mathrm{NL}}^{(\mathrm{local})}\ \zeta_g^2 + \ldots, \tag{123}$$

where we have expanded the curvature perturbation to the non linear order. Here ζ_g is the Gaussian part of the curvature perturbation. The next term is the leading non linear correction to the curvature perturbation where the factor of $\frac{3}{5}$ arises from the transfer potential during inflation to the curvature perturbation. The $f_{\mathrm{NL}}^{(\mathrm{local})}$ parameter is the amplitude of the non Gaussianity of the curvature perturbations. "Local" here refers to the type of non-Gaussianity, where the non-linear response is localized in real space, meaning that the non-linear interaction depends only on the value of ζ_g at a given point in space and not on its behavior over a region. The correlations between the Fourier modes which were not present initially are introduced by the quadratic term ζ_g^2. Therefore the presence of connected higher-order moments gives rise to non-Gaussianity from the non-linearity of the curvature perturbation. This expansion then gives us a bispectrum of the following form:

$$B_{\zeta}^{(\mathrm{local})}(k_1,k_2,k_3) = \frac{6}{5} f_{\mathrm{NL}}^{(\mathrm{local})} \Big(P_{\zeta}(k_1) P_{\zeta}(k_2) + P_{\zeta}(k_2) P_{\zeta}(k_3) + P_{\zeta}(k_3) P_{\zeta}(k_1) \Big) \tag{124}$$

$$= \frac{6}{5} f_{\mathrm{NL}}^{(\mathrm{local})} A_s^2 \left(\frac{1}{k_1^{4-n_s} k_2^{4-n_s}} + \frac{1}{k_2^{4-n_s} k_3^{4-n_s}} + \frac{1}{k_3^{4-n_s} k_1^{4-n_s}} \right), \tag{125}$$

Primordial fluctuations of this type are typically produced during inflation from an isocurvature field whose fluctuations are converted into an adiabatic one after inflation. There are several models which produce local-type non-Gaussianity. These models include the modulated reheating model [62], the mixed inflaton and modulated reheating model [88], the curvaton model [65, 130, 139], the mixed inflaton and curvaton model [70, 87, 107, 108, 140, 141], the inhomogeneous end of hybrid inflation model [7, 34, 36, 128, 155], the multi-brid inflation model [146, 156], multi-field inflation models [42, 43, 161], the ungaussiton model [38, 165], the multi-curvaton model [15, 50, 167], the modulated curvaton model [51, 166], the model with an inhomogeneous cosmological phase transition [135], the model with an inhomogeneous end of thermal inflation [90], modulated trapping [26, 105], the model with modulated decay of the curvaton [14, 64, 106], the hybrid curvaton-modulaton model [106], the ekpyrotic scenario models [92, 104, 162], and some single-field inflation models with an early non-attractor phase [47, 145] amongst others.

We also have local-type non-Gaussianity in models where isocurvature fluctuations are present. But in this case, another non-linearity parameter has to be introduced to deal with isocurvature fluctuations, which is currently not of primary interest for the thesis.

Equilateral Type

Equilateral-type non-Gaussianity arises in scenarios where single-field inflation models have a non-cannonical kinetic term or interactions with higher-derivative operators.

involving general single-field inflation models with a non-canonical kinetic term or interactions with higher-derivative operators. An observational constraint on this type of non-Gaussianity is typically obtained by adopting a template with the following form:

$$B_\zeta^{(\text{equil})}(k_1,k_2,k_3) = \frac{3}{5} f_{\text{NL}}^{(\text{equil})}[-3P_\zeta\,(k_1)P_\zeta\,(k_2) - 2P_\zeta^{2/3}\,(k_1)\,P_\zeta^{2/3}\,(k_2)\,P_\zeta^{2/3}\,(k_3) + 6P_\zeta^{1/3}\,(k_1)\,P_\zeta^{2/3}\,(k_2)\,P_\zeta\,(k_3) + (5 \text{ permutations})] \quad (126)$$

Here, $P_\zeta(k)$ denotes the power spectrum of the curvature perturbation ζ, and $f_{\rm NL}^{(\rm equil)}$ represents the non-linearity parameter specific to equilateral-type non-Gaussianity. The permutations involve cyclic permutations of the momenta (k_1, k_2, k_3).

Trispectrum

In the previous we went through the properties of the bispectrum. In this section we shall go over the trispectrum, discussing the local type.

Local Type

The curvature perturbations for the local type trispectrum can be further expanded as:

$$\zeta = \zeta_g + \frac{3}{5} f_{\rm local}^{(\rm NL)} \zeta_g^2 + \left(\frac{3}{5}\right)^2 g_{\rm local}^{(\rm NL)} \zeta_g^3 + \dots \tag{127}$$

from which we can compute the trispectrum:

$$T_\zeta(k_1, k_2, k_3, k_4) = \tau_{\rm local}^{(\rm NL)} [P_\zeta(k_{13}) P_\zeta(k_3) P_\zeta(k_4) + 11 \text{ perms.}] + \frac{54}{25} g_{\rm local}^{(\rm NL)} [P_\zeta(k_2) P_\zeta(k_3) P_\zeta(k_4) + 3 \text{ perms.}] \tag{128}$$

where $k_{13} = |k_1 + k_3|$ and $\tau_{\rm local}^{(\rm NL)}$ and $g_{\rm local}^{(\rm NL)}$ are non-linearity parameters characterizing the size of the trispectrum. Regarding $g_{\rm NL}$, most models have a particular relation between $f_{\rm NL}$ and $g_{\rm NL}$, which can be written for simplicity as:

Table 3. Types of $g_{\rm NL}$ in Relation to $f_{\rm NL}$

Type of $g_{\rm NL}$	**Relation to $f_{\rm NL}$**
"linear $g_{\rm NL}$" type	$\lvert g_{\rm NL}\rvert \sim \lvert f_{\rm NL}\rvert$
"suppressed $g_{\rm NL}$" type	$\lvert g_{\rm NL}\rvert \sim$ (suppression factor) $\times \lvert f_{\rm NL}\rvert$
"enhanced $g_{\rm NL}$" type	$\lvert g_{\rm NL}\rvert \sim \lvert f_{\rm NL}\rvert^n$ (with $n > 1$)

The relation between $f_{\rm NL}$ and $g_{\rm NL}$, however, actually depends on the model parameters, the above classification depicts how large $g_{\rm NL}$ can be. Therefore it is critical to quantify these non-

Gaussianities through observables, following the techniques to be discussed in the next section. An extensive review of primordial non-Gaussianity can be found in these detailed works [2, 19, 22–24, 28, 35, 37, 45, 48, 49, 52, 57, 67–69, 85, 112, 113, 131, 133, 136, 168, 177]

STATISTICAL METHODS FOR PARAMETER ESTIMATION

Statistical inferencing can be divided into two main approaches, namely frequentist and Bayesian. In this thesis we will work with the Bayesian approach for the parameter estimation of the f_{NL}^{local} parameter. Taking θ as the parameter of interest in the model. For the Bayesian approach we specify a joint probability distribution for the parameters and the data, which is known as the posterior distribution based on Baye's theorem.

$$P(\theta|\text{data}) = \frac{P(\text{data}|\theta)P(\theta)}{P(\text{data})} \tag{129}$$

Where $P(\theta)$ is defined as the prior distribution which describes our initial beliefs regarding the parameters, while the likelihood function which represents the probability of observing the data given the parameters if defined by $P(\text{data}|\theta)$. For many cases, we work with the assumption that the data parameters are independent and identically distributed which allows us to write the likelihood as a product of individual probabilities as follows:

$$P(\text{data}|\theta) = \prod_{i=1}^{N} P(\text{data}_i|\theta) \tag{130}$$

Uniform, Gaussian or exponential distributions are usually taken as the common choice for priors. Updated beliefs about the parameters after observing the data are denoted by the posterior distribution, denoted by $P(\theta|\text{data})$. The posterior distribution is obtained by combining the likelihood and prior using Bayes' theorem. Once we have the posterior distribution, we can perform parameter estimation using MCMC methods. MCMC algorithms, such as the Metropolis-Hastings algorithm or Gibbs sampling, generate samples from the posterior distribution, which allow us to approximate the posterior's properties and estimate the parameters of interest. In addition to Bayesian methods, chi-square analysis is a commonly used technique for parameter estimation and model fitting. It involves minimizing the chi-square statistic, defined as:

$$\chi^2 = \sum_{i=1}^{N} \frac{(y_i - f(x_i))^2}{\sigma_i^2} \tag{131}$$

where y_i are the observed data points, $f(x_i)$ are the model predictions, and σ_i are the uncertainties associated with the data points. The lower the value of the chi square statistic, the better the fit for the model. Implementing both the techniques together is really beneficial cause we can have a two pronged attack where we can do parameter estimation and check the fit of the model. We will now look at the implementation in the next section

MONTE CARLO MARKOV CHAINS AND CHI-SQUARE ANALYSIS

Setup And Methodology

The model for small field inflation was taken for the mcmc and chi square analysis of the thesis. The potential of the small field inflation model is given as follows:

$$V(\phi) = M^4 \left[1 - \left(\frac{\phi}{\mu}\right)^p\right] \tag{132}$$

The given model is a two parameter model depending on the value of μ which is the vaccum energy scale and the power law exponent p. The values of M^4 which the normalisation and the scalar field ϕ are based on the upper bound of μ. Therefore we can use the slow roll approximation and the general equation for f_{NL}^{local} for single field inflation models which is given as:

$$f_{NL}^{local} = -5/12\,(n_s - 1) \tag{133}$$

which was given by Juan Maldacena in his brilliant work [131]. I took the observational constraints for n_s and f_{NL}^{local} from the planck 2018 releases for cosmological parameters and primordial non-Gaussianity [5,6]. I then used the numpy open source library for python [82] to define the priors and likelihoods for the observational distributions. I used the emcee ensemble sampler to perform the mcmc sampling. A complete overview of the emcee library can be found here [71].

The methodology was self implemented along with some inspiration from these two works [58,93]. Finally after the analysis the plots and distributions were plotted using the matplotlib and corner libraries in python. The implemented code is discussed in the next section.

Python Code

As discussed in the previous section, the complete implementation of the methodology through python code is given below.

```python
import numpy as np
import emcee
import matplotlib.pyplot as plt
import corner

# Observational data for ns and f_NL from Planck 2018 Results
fnl_obs = -0.9
fnl_std = 5.1
ns_obs = 0.9649
ns_std = 0.0042

# Model definition for Small Field Inflation
def V(phi, V0, mu, p):
    return V0 * (1 - (phi / mu)**p)

# First derivative of the field potential
def V_prime(phi, V0, mu, p):
    return -p * V0 * (phi / mu)**(p-1) / mu

# Second derivative of the field potential
def V_double_prime(phi, V0, mu, p):
    return -p * (p-1) * V0 * (phi / mu)**(p-2) / mu**2

# First slow roll parameter
```

```python
def epsilon(phi, V0, mu, p):
    return 0.5 * (V_prime(phi, V0, mu, p) / V(phi, V0, mu, p))**2

# Second slow roll parameter
def eta(phi, V0, mu, p):
    return V_double_prime(phi, V0, mu, p) / V(phi, V0, mu, p)

# Function to calculate the scalar spectral index
def model_ns(phi, V0, mu, p):
    eps = epsilon(phi, V0, mu, p)
    et = eta(phi, V0, mu, p)
    return 1 - 6 * eps + 2 * et

# Uniform priors for ns for mcmc analysis
def log_prior(phi, V0, mu, p):
    if 1e-9 <= V0 <= 1e-3 and 1 <= mu <= 50 and 1 <= p <= 10 and 0 <= phi <= mu:
        return 0  # log of 1 for uniform distribution
    else:
        return -np.inf # Out of bounds


# Likelihood function for ns for the mcmc analysis
def log_likelihood_ns(phi, V0, mu, p):
    ns_model = model_ns(phi, V0, mu, p)
    return -0.5 * ((ns_model - ns_obs)**2 / ns_std**2)

# Probability function for the mcmc analysis
def log_probability_ns(params):
    phi, V0, mu, p = params
    lp = log_prior(phi, V0, mu, p)
    if not np.isfinite(lp):
        return -np.inf
```

```
    ll = log_likelihood_ns(phi, V0, mu, p)
    return lp + ll

# MCMC setup
nwalkers, ndim, nsteps = 50, 4, 5000
start_guesses = np.random.rand(nwalkers, ndim) * [mu, 1e-3 - 1e-9, 48, 9]
    + [0, 1e-9, 1, 1]

# Initialize sampler
sampler_ns = emcee.EnsembleSampler(nwalkers, ndim, log_probability_ns)
print("Running MCMC for ns...")
state_ns = sampler_ns.run_mcmc(start_guesses, nsteps, progress=True)
samples_ns = sampler_ns.get_chain(flat=True)

# Calculate fnl using ns distribution
def calculate_fnl_from_ns_distribution(ns_samples):
    ns_samples_array = np.array(ns_samples)  # Convert list to numpy array
    return 5/12 * (1 - ns_samples_array)

# Calculate n_s for all samples
ns_samples = np.array([model_ns(phi, V0, mu, p) for phi, V0, mu, p in
    samples_ns])

# Calculate the mean and standard deviation of n_s
ns_mean = np.mean(ns_samples)
ns_std = np.std(ns_samples)

# Calculate fnl using ns distribution
fnl_samples_from_ns = calculate_fnl_from_ns_distribution(ns_samples)
fnl_mean_from_ns = np.mean(fnl_samples_from_ns)
fnl_std_from_ns = np.std(fnl_samples_from_ns)

# Calculate the chi square statistic
```

```
chi_square = np.sum((fnl_mean_from_ns - fnl_obs)**2 / fnl_std**2)

# Print all the results
print(f"Mean n_s after MCMC analysis for small field inflation: {ns_mean
    :.5f}")
print(f"Standard Deviation of n_s after MCMC analysis for small field
    inflation: {ns_std:.5f}")
print(f"Mean f_NL calculated from n_s samples for small field inflation: {
    fnl_mean_from_ns:.5f}")
print(f"Standard Deviation of f_NL from n_s samples for small field
    inflation: {fnl_std_from_ns:.5f}")
print(f"Chi-square statistic for f_NL: {chi_square:.5f}")


# Assuming samples_ns and samples_fnl are numpy arrays containing MCMC
    samples.

# Corner plot for n_s
fig_ns = corner.corner(
    samples_ns,
    labels=[r"$\phi$", r"$V_0$", r"$\mu$",r"p"],
    title_kwargs={'fontsize': 12},
    label_kwargs={'fontsize': 12},
    hist_kwargs={'density': True},
    show_titles=True,
    quantiles=[0.16, 0.5, 0.84],
    title_fmt='.4f',
    plot_density=True,
    plot_datapoints=False,
    fill_contours=True
)
fig_ns.suptitle("Corner plots for $n_s$ parameters", fontsize=14, y=1.05)
     # Adjust y for better placement if needed
```

```
plt.show()

# Plotting posterior distribution of ns
plt.figure(figsize=(8, 6))
plt.hist(ns_samples, bins=30, range=(0.9, 1.0), alpha=0.6, color='blue', edgecolor='black')
plt.title('Posterior Distribution of $n_s$ for Small Field Inflation Model', fontsize=16)
plt.xlabel('$n_s$', fontsize=14)
plt.ylabel('Frequency', fontsize=14)
plt.xticks(fontsize=12)
plt.yticks(fontsize=12)
plt.grid(True, linestyle='--', alpha=0.5)
plt.tight_layout()
plt.savefig('posterior_ns_distribution.png')  # Save the figure if needed
plt.show()

# Plotting posterior distribution of fnl
plt.figure(figsize=(8, 6))
plt.hist(fnl_samples_from_ns, bins=30, range=(0, 0.004),alpha=0.6, color='blue', edgecolor='black')
plt.title('Posterior Distribution of $f_{NL}$ for Small Field Inflation Model', fontsize=16)
plt.xlabel('$f_{NL}$', fontsize=14)
plt.ylabel('Frequency', fontsize=14)
plt.xticks(fontsize=12)
plt.yticks(fontsize=12)
plt.grid(True, linestyle='--', alpha=0.5)
plt.tight_layout()
plt.savefig('posterior_fnl_distribution.png')  # Save the figure if needed
plt.show()
```

Results And Conclusions

From the above implementation of the code we got the following results:

Table 4. Statistical Results for Small Field Inflation

Metric	**Value**
n_s	0.96441 ± 0.18324
f_{NL}^{local}	0.01483 ± 0.07635
Chi-square statistic for f_{NL}^{local}	0.03218

As we can see from the results, the estimated f_{NL}^{local} for single field inflation models, obtained through the MCMC analysis matches the theoretical predictions and is very less than $\mathcal{O}(1)$. We can also infer from the chi square value that the model is not a good fit for the observational data thus ruling out this model. The corner and the posterior distribution plots are given below.

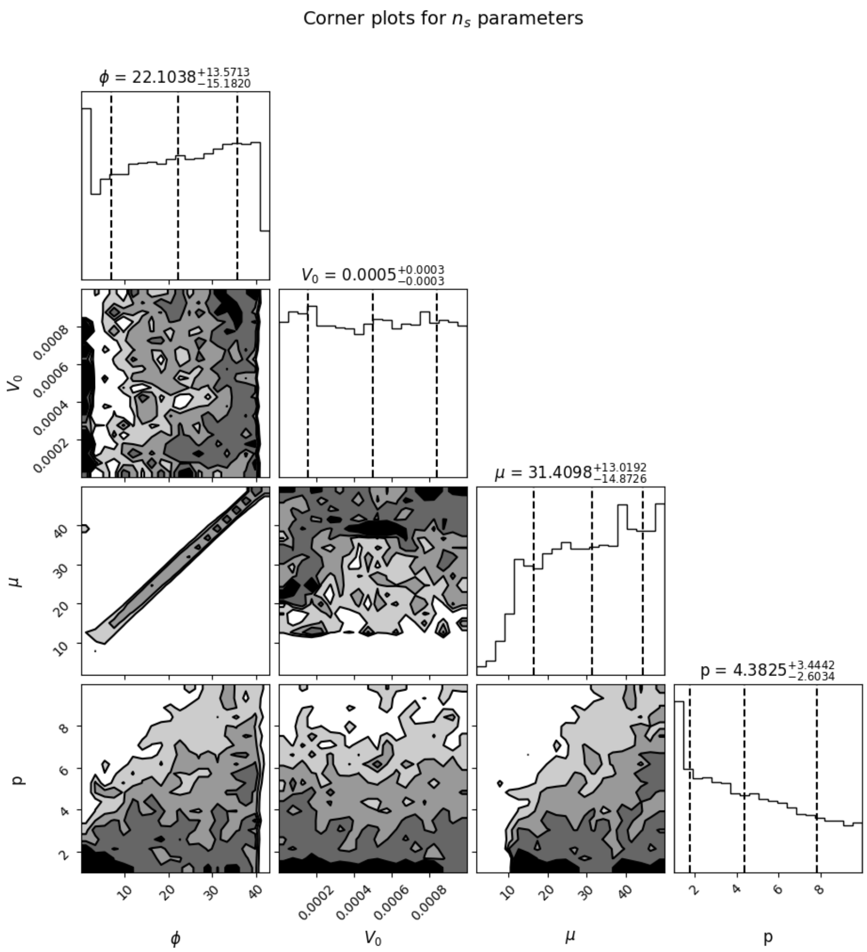

Figure 4. Corner plots for n_s using mcmc

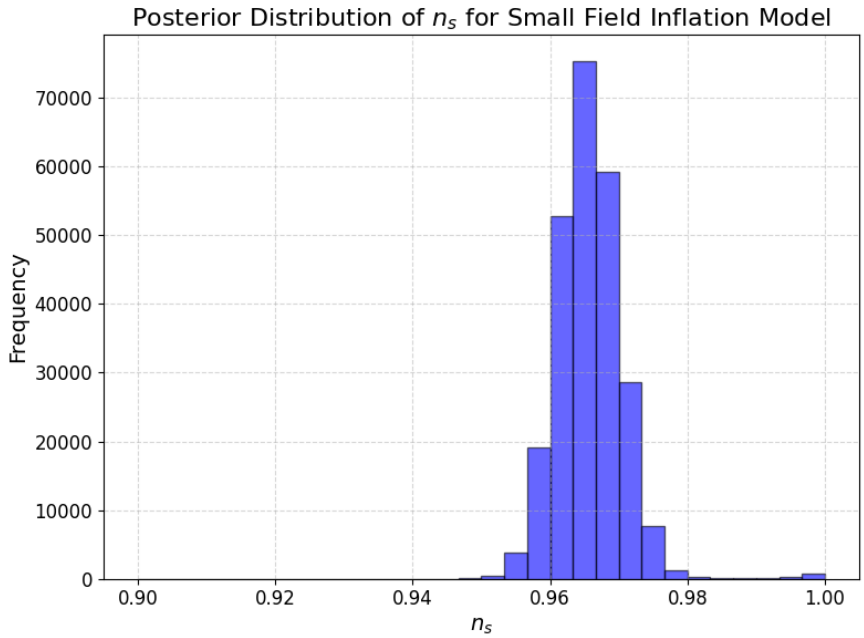

Figure 5. Posterior Distribution for n_s

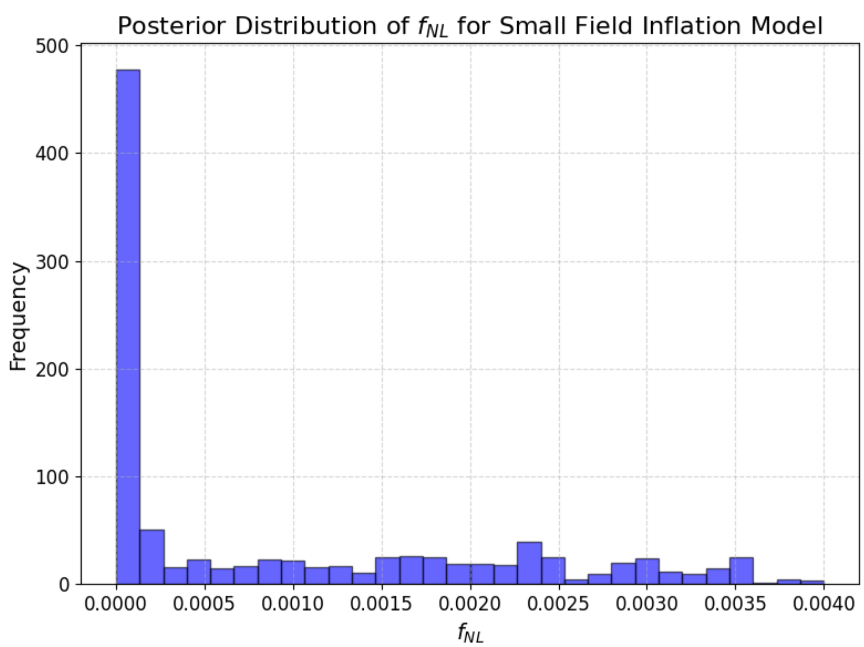

Figure 6. Posterior Distribution for f_{NL}^{local}

Therefore from the analysis we can conclude that MCMC sampling techniques can be used for the estimation of parameters such as f_{NL}^{local} which help us in better understanding the dynamics of the early universe and self consistency of several inflationary models.

Further Scope

The analysis looks at the local non-linearity parameter of the bi-spectrum for primordial non-Gaussianity for small field inflation model. The analysis can be extended further by considering the equilateral and flat types as well along with looking at the local and equilateral types for the trispectrum for other single field models. One can also take single field models with non canonical kinetic terms and/or models with slow roll violation. A further advanced analysis can be done on multifield models taking into account spectator fields and isocurvature perturbation models apart from the normal adiabatic modes. Non GR inflationary models can also be analysed with exotic scenarios. Performing the MCMC and chi square analysis for these models can provide a much more better view of the early universe dynamics and better constraints regarding self consistency for these models and lead us to a much better description of our universe.

APPENDIX 1: GAUGE CHOICES

Another way of avoiding the fictitious gauge modes problem is to fix a gauge and keep track of all perturbations in both the metric and the matter. A particular gauge can be used for a specific purpose. Below I will describe in detail the gauges used in the thesis and some other useful gauges.

Newtonian Gauge

The Newtonian Gauge is called such because it reduces to Newtonian gravity in the small scale limit. It is useful because we can get the algebraic relations between metric and stress-energy perturbations. Newtonian gauge is defined with the following constraints on the scalar perturbations.

$$E = B = 0 \tag{A.1}$$

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Psi)\delta_{ij}dx^i dx^j \tag{A.2}$$

We have the Einstein equations as

$$\begin{aligned} 3H(\dot{\Psi}+H\Phi)+\frac{k^2}{a^2}\Psi &= -4\pi G\,\delta\rho && \text{(A.3)}\\ \dot{\Psi}+H\Phi &= -4\pi G\,\delta q && \text{(A.4)}\\ \ddot{\Psi}+3H\dot{\Psi}+H\dot{\Phi}+(3H^2+2\dot{H})\Phi &= 4\pi G\left(\delta p-\frac{2}{3}k^2\delta\Sigma\right) && \text{(A.5)}\\ \frac{\Psi-\Phi}{a^2} &= 8\pi G\,\delta\Sigma. && \text{(A.6)} \end{aligned}$$

And the continuity equations are

$$\begin{aligned} \dot{\delta\rho}+3H(\delta\rho+\delta p) &= \frac{k^2}{a^2}\delta q+3(\bar{\rho}+\bar{p})\dot{\Psi}, && \text{(A.7)}\\ \dot{\delta q}+3H\delta q &= -\delta p+\frac{2}{3}k^2\delta\Sigma-(\bar{\rho}+\bar{p})\Phi. && \text{(A.8)} \end{aligned}$$

Comoving Gauge

Comoving gauge is characterised by the vanishing of the scalar momentum density,

$$\delta q = 0, \qquad E = 0. \tag{A.9}$$

We also conventionally set $\Phi \equiv \mathcal{R}$ in this gauge.

The Einstein Equations are

$$
\begin{aligned}
3H(-\dot{\mathcal{R}}+H\Phi)+\frac{k^2}{a^2}[-\mathcal{R}-aHB] &= -4\pi G\,\delta\rho && \text{(A.10)}\\
-\dot{\mathcal{R}}+H\Phi &= 0 && \text{(A.11)}\\
-\ddot{\mathcal{R}}-3H\dot{\mathcal{R}}+H\dot{\Phi}+(3H^2+2\dot{H})\Phi &= 4\pi G\left(\delta p-\frac{2}{3}k^2\delta\Sigma\right) && \text{(A.12)}\\
(\partial_t+3H)B/a+\frac{\mathcal{R}+\Phi}{a^2} &= -8\pi G\,\delta\Sigma. && \text{(A.13)}
\end{aligned}
$$

The continuity equations are

$$
\begin{aligned}
\dot{\delta\rho}+3H(\delta\rho+\delta p) &= (\bar{\rho}+\bar{p})[-3\dot{\mathcal{R}}+k^2B/a]. && \text{(A.14)}\\
0 &= -\delta p+\frac{2}{3}k^2\delta\Sigma-(\bar{\rho}+\bar{p})\Phi. && \text{(A.15)}
\end{aligned}
$$

Spatially-flat Gauge

We use the spatially flat gauge for computing inflationary perturbations where we consider the following constraints

$$
\Psi = E = 0. \tag{A.16}
$$

During inflation all scalar perturbations are then described by $\delta\phi$. The Einstein Equations are

$$
\begin{aligned}
3H^2\Phi+\frac{k^2}{a^2}[-aHB)] &= -4\pi G\,\delta\rho && \text{(A.17)}\\
H\Phi &= -4\pi G\,\delta q && \text{(A.18)}\\
H\dot{\Phi}+(3H^2+2\dot{H})\Phi &= 4\pi G\left(\delta p-\frac{2}{3}k^2\delta\Sigma\right) && \text{(A.19)}\\
(\partial_t+3H)B/a+\frac{\Phi}{a^2} &= -8\pi G\,\delta\Sigma. && \text{(A.20)}
\end{aligned}
$$

The continuity equations are

$$
\begin{aligned}
\dot{\delta\rho}+3H(\delta\rho+\delta p) &= \frac{k^2}{a^2}\delta q+(\bar{\rho}+\bar{p})[k^2B/a], && \text{(A.21)}\\
\dot{\delta q}+3H\delta q &= -\delta p+\frac{2}{3}k^2\delta\Sigma-(\bar{\rho}+\bar{p})\Phi. && \text{(A.22)}
\end{aligned}
$$

These gauges are very useful for different applications pertaining to perturbation theory.

APPENDIX 2: PRIMORDIAL BLACKHOLES, THE SWAMPLAND AND EXOTIC INFLATIONARY MODELS

PBHs form when density fluctuations are comparable to $\mathcal{O}(1)$ at the horizon crossing. Hence, for PBHs to constitute dark matter, one requires a large amplification of the inflationary power spectrum between the cosmic microwave background (CMB) and the PBH mass scales. The swampland criteria have direct consequences for formation of PBHs and dark matter. In the paper by Masahiro Kawasaki and Volodymyr Takhistov [91], we see that the swampland conjectures as originally proposed, are incompatible with formation of PBHs that can constitute dark matter in the context of single field inflation models which highlighted the importance of placing restrictions on the behavior of the scalar fields in EFTs with regards to the structure formation in early universe and the bigger implications for dark matter. In our upcoming paper we extend the work done by Kawasaki and Takhistov by using several more realistic inflationary models such as warm inflation models, ultra slow roll models (which are found to be in support of PBHs as dark matter candidates), constant roll models, etc. We also incorporate modified gravity models and Non GR models such as Chern Simons theory. Taking these models, we then constrain these free parameters or specific potentials in order to be consistent with both the swampland and the observational requirements, thereby arriving at the best conclusions of whether or not PBHs can be supported for these different models.

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